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Non-Pfaffian quasi-bi-Hamiltonian systems with two degrees of freedom

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Abstract. In the case of two degrees of freedom, a (non-Pfaffian) quasi-bi-Hamiltonian system with a separable integrating factor is presented (in terms of Nijenhuis coordinates) and its separability is proved. Indications are given in the case of a general integrating factor.

1. Introduction

Magri in 1977 [12] introduced a very interesting notion which explains the complete integrability [1–3] of certain Hamiltonian systems. Indeed, when a Hamiltonian vector field is also Hamiltonian for a second Poisson structure compatible with the previous one it is completely integrable under suitable conditions [12, 13]. Many classical dynamical systems in both finite and infinite dimensions are known to have such bi-Hamiltonian formulations [12, 13], and recent papers [11, 16] provide methods to construct a second compatible structure in certain examples. Nevertheless, it remains very difficult to exhibit [15] a bi-Hamiltonian structure for a given vector field. Moreover, the existence of such a structure on a whole neighbourhood of a Liouville torus, imposes for a large class of Hamiltonians very drastic conditions [4–6, 8].

For these reasons, we recently introduced a weaker structure called *quasi-bi-Hamiltonian structure* (QBHS) [7, 14]. We only ask for a Hamiltonian vector to be, after multiplication by some function (called the *integrating factor*) Hamiltonian for a compatible second Poisson bracket. This kind of structure is easier to obtain in explicit examples [15], and has in addition interesting properties concerning integrability [7, 14]. Moreover, the above strong conditions are relaxed [7, 14].

In this paper we study a QBHS with two degrees of freedom with a special type of integrating factor. Precisely, if a vector field X is Hamiltonian of Hamiltonian H for a first structure, and if ρX is Hamiltonian of Hamiltonian F for a compatible second one, we know that $\rho = -\lambda_1\lambda_2 f(H, F)$ [7, 14], where λ_1, λ_2 denote eigenvalues of the Nijenhuis $(1, 1)$-tensor field defined by the two compatible Poisson brackets.

Here, we are interested in the special case $\rho = -\lambda_1\lambda_2 f_1(H) f_2(F)$. We first prove that canonical coordinates associated to these eigenvalues (called *Nijenhuis coordinates*) also allow us to separate the Hamilton–Jacobi equation as in the Pfaffian case. Next, we give some results about the so-called Levi-Civita operator [3, 9] for a general integrating factor. Finally, we attempt to touch on the separability of the Hamilton–Jacobi equation for general integrating factors.

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2. Reminder about QBHS

Definition 1 ([7, 14]). A Hamiltonian system (M, C_0, H) where M is a manifold of an even dimension endowed with a non-degenerate Poisson structure C_0 and $H \in C^{\infty}(M, \mathbb{R})$, is said to have a QBHS if it exists such that:

(i) a Poisson structure C compatible with C_0, i.e. $C + C_0$ is also a Poisson structure,

(ii) a function $F \in C^{\infty}(M, \mathbb{R})$, and

(iii) a non-vanishing function $\rho \in C^{\infty}(M, \mathbb{R})$ (called the *integrating factor*) so that the relation:

$$C_0(., H) = \frac{1}{\rho} C(., F) \tag{1}$$

is verified.

The 6-tuple (M, C_0, H, C, F, ρ) is called a *quasi-bi-Hamiltonian system*. We remark that $C_0(F, H) = \frac{1}{\rho} C(F, F) = 0$; so that, F is a first integral of the Hamiltonian field $C_0(., H)$.

Definition 2. We say that a quasi-bi-Hamiltonian system (M, C_0, H, C, F, ρ) is *real decomposable* [5] if the operator $J = CC_0{}^{-1}$ (connecting the two compatible Poisson structures C and C_0) has the maximum number $(= \frac{1}{2} \dim\ M)$ of distinct real eigenvalues at each point (so that J is diagonalizable).

In the following, we assume $\dim M = 4$.

Proposition 1 ([7, 14]). Let (M, C_0, H, C, F, ρ) be a QBHS. If H and F are functionally independent, then $\frac{\rho^2}{\det J}$ is a function $f(H, F)$ of H and F.

Remark 1.

(i) Note that the converse of proposition 1 is false (see [14] for the proof). In [7, 14] we have defined and studied a particular case (called Pfaffian QBHS) where $\frac{\rho^2}{\det J} = 1$, i.e. $f(H, F) = 1$, and H and F are not necessarily functionally independent. So we have $\rho = -\lambda_1 \lambda_2$, where λ_i $(i = 1, 2)$ are eigenvalues of J.

(ii) Since C_0 and C are compatible, the eigenvalues λ_i $(i = 1, 2)$ of the operator $J = CC_0{}^{-1}$ are in involution with respect to the C_0 and C [12, 13]. We suppose that they are real, distinct (i.e. J is real decomposable) and functionally independent. Hence, we can complete (λ_1, λ_2) [7] by functions $(p_{\lambda_1}, p_{\lambda_2})$ so that $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$ are canonical coordinates. They are called *Nijenhuis coordinates* [7].

We also recall below an important result concerning the study of a Pfaffian QBHS with respect to these canonical coordinates.

Proposition 2 ([7, 14]). Let $(M, C_0, H, C, F, -\lambda_1\lambda_2)$ be a Pfaffian QBHS. In the Nijenhuis coordinates, the Hamiltonian H and the second Hamiltonian F take the following forms

$$H = \frac{H_1(\lambda_1, p_{\lambda_1}) - H_2(\lambda_2, p_{\lambda_2})}{\lambda_1 - \lambda_2} \tag{2}$$

$$F = \frac{-\lambda_2 H_1(\lambda_1, p_{\lambda_1}) + \lambda_1 H_2(\lambda_2, p_{\lambda_2})}{\lambda_1 - \lambda_2}. \tag{3}$$

Definition 3. We say that the pair (H, F) of functions satisfying (2) and (3) presents a *Pfaffian Gantmacher form*.

Remark 2. The condition (2) implies that the Nijenhuis coordinates separate Hamilton–Jacobi equation associated with the system. In fact, the Levi-Civita operator [3, 9] denoted by LV applied to H vanishes, where

$$LV = \frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}\frac{\partial^2}{\partial p_{\lambda_1}\partial p_{\lambda_2}} - \frac{\partial}{\partial p_{\lambda_1}}\frac{\partial}{\partial\lambda_2}\frac{\partial^2}{\partial p_{\lambda_2}\partial\lambda_1} - \frac{\partial}{\partial p_{\lambda_2}}\frac{\partial}{\partial\lambda_1}\frac{\partial^2}{\partial p_{\lambda_1}\partial\lambda_2} + \frac{\partial}{\partial p_{\lambda_1}}\frac{\partial}{\partial p_{\lambda_2}}\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} \tag{4}$$

provided that $\frac{\partial H}{\partial p_{\lambda_i}}\frac{\partial H}{\partial\lambda_i} \neq 0$ for $i = 1, 2$.

In the following section, we study a QBHS (M, C_0, H, C, F, ρ) with a general integrating factor. At first, we take interest in the case where the integrating factor $\rho = -\lambda_1\lambda_2 f(H, F)$ can be written $\rho = -\lambda_1\lambda_2 f_1(H) f_2(F)$, where f_1, f_2 are two non-vanishing functions in $C^\infty(M, \mathbb{R})$, called here *separable form*. After that, we give some results about the general case (without restriction on the function $f(H, F)$).

3. The QBHS with a separable integrating factor $\rho = -\lambda_1\lambda_2 f_1(H) f_2(F)$

Results concerning this case are stated in the following proposition.

Proposition 3. Let (M, C_0, H, C, F, ρ) be a QBHS. If the integrating factor has the separable form $\rho = -\lambda_1\lambda_2 f_1(H) f_2(F)$, then in Nijenhuis coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$, the functions H and F take the following forms:

$$H = h\left(\frac{H_2(\lambda_2, p_{\lambda_2}) - H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right) \tag{5}$$

$$F = g\left(\frac{-\lambda_1 H_2(\lambda_2, p_{\lambda_2}) + \lambda_2 H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right) \tag{6}$$

where h and g are functions deduced from f_i.

Moreover, the Hamilton–Jacobi equation associated with the system is separable in these coordinates.

Proof. First, we recall [7, 14] that in the Nijenhuis coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$, the two Poisson structures C_0 and C can be written respectively

$$C_0 = \frac{\partial}{\partial\lambda_1} \wedge \frac{\partial}{\partial p_{\lambda_1}} + \frac{\partial}{\partial\lambda_2} \wedge \frac{\partial}{\partial p_{\lambda_2}} \tag{7}$$

$$C = \lambda_1\frac{\partial}{\partial\lambda_1} \wedge \frac{\partial}{\partial p_{\lambda_1}} + \lambda_2\frac{\partial}{\partial\lambda_2} \wedge \frac{\partial}{\partial p_{\lambda_2}}. \tag{8}$$

Therefore, relation (1) defining a QBHS with integrating factor $\rho = -\lambda_1\lambda_2 f_1(H) f_2(F)$, can be defined explicitly by equations:

$$\frac{\partial H}{\partial\lambda_1} = -\frac{1}{\lambda_2 f_1 f_2}\frac{\partial F}{\partial\lambda_1} \tag{9a}$$

$$\frac{\partial H}{\partial\lambda_2} = -\frac{1}{\lambda_1 f_1 f_2}\frac{\partial F}{\partial\lambda_2} \tag{9b}$$

$$\frac{\partial H}{\partial p_{\lambda_1}} = -\frac{1}{\lambda_2 f_1 f_2}\frac{\partial F}{\partial p_{\lambda_1}} \tag{9c}$$

$$\frac{\partial H}{\partial p_{\lambda_2}} = -\frac{1}{\lambda_1 f_1 f_2}\frac{\partial F}{\partial p_{\lambda_2}}. \tag{9d}$$

A straightforward integration of (9) leads to the functions

$$F_1(H) = \frac{H_2(\lambda_2, p_{\lambda_2}) - H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1} \tag{10}$$

$$F_2(F) = \frac{-\lambda_1 H_2 + \lambda_2 H_1}{\lambda_2 - \lambda_1} \tag{11}$$

where $F_1(H)$ (resp. $F_2(F)$) is a primitive of $f_1(H)$ (resp. $f_2(F)^{-1}$) and $H_i(\lambda_i, p_{\lambda_i})$ $(i = 1, 2)$ are arbitrary functions.

From the inverse function theorem, we obtain

$$H = h\left(\frac{H_2(\lambda_2, p_{\lambda_2}) - H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right) \tag{12}$$

$$F = g\left(\frac{-\lambda_1 H_2(\lambda_2, p_{\lambda_2}) + \lambda_2 H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right) \tag{13}$$

where h (resp. g) is the inverse function of F_1 (resp. F_2).

Moreover, a straightforward calculation leads to

$$LV(H) = \frac{\mathrm{d}h}{\mathrm{d}\tilde{H}} LV(\tilde{H}) \tag{14}$$

where we denote

$$\tilde{H} = \frac{H_2(\lambda_2, p_{\lambda_2}) - H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}.$$

From remark 2, $LV(\tilde{H}) = 0$. Then $LV(H) = 0$. □

It is natural to study the converse of proposition 3. The following proposition provides a result of this study.

Proposition 4. Let $H_1(q_1, p_1)$, $G_1(q_1, p_1)$, $H_2(q_2, p_2)$, $G_2(q_2, p_2)$ be arbitrary functions belonging to $C^\infty(\mathbb{R}^2, \mathbb{R})$. Denote

$$H = h\left(\frac{H_2 - H_1}{G_2 - G_1}\right) \tag{15}$$

$h \in C^\infty(\mathbb{R}, \mathbb{R})$ and

$$C_0 = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2} \tag{16}$$

the canonical Poisson structure in $\mathbb{R}^4$. Then the Hamiltonian system $(\mathbb{R}, C_0, H)$ admits a QBHS defined by

$$C_0(., H) = \frac{1}{\rho} C(., F)$$

where

$$\text{(i)} \qquad F = g\left(\frac{-q_1 H_2 + q_2 H_1}{G_2 - G_1}\right) \tag{17}$$

with $g \in C^\infty(\mathbb{R}, \mathbb{R})$

$$\text{(ii)} \qquad \rho = -G_1 G_2 f_1(H) f_2(F) \tag{18}$$

with

$$f_1(H) = \frac{1}{\frac{\partial h}{\partial H'} \circ h^{-1}}(H)$$

and

$$f_2(F) = \left(\frac{\partial g}{\partial F'} \circ g^{-1}\right)(F)$$

we denote

$$H' = \frac{H_2 - H_1}{G_2 - G_1}$$

and

$$F' = \frac{-q_1 H_2 + q_2 H_1}{G_2 - G_1} \tag{19}$$

$$\text{(iii)} \qquad C = G_1 \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + G_2 \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2}.$$

Proof. Let $H = h(H')$. Taking $F = g(F')$, a straightforward calculation leads to

$$\frac{\partial H}{\partial q_1} = -\frac{1}{G_2} \frac{\frac{\partial h}{\partial H'}}{\frac{\partial g}{\partial F'}} \frac{\partial F}{\partial q_1}. \tag{20}$$

Applying the inverse function theorem to the expressions (5) of H and (6) of F, (20) becomes

$$\frac{\partial H}{\partial q_1} = -\frac{1}{G_2} \frac{(\frac{\partial h}{\partial H'} \circ h^{-1})(H)}{(\frac{\partial g}{\partial F'} \circ g^{-1})(F)} \frac{\partial F}{\partial q_1}. \tag{21}$$

Setting, $f_1(H) = \frac{1}{(\frac{\partial h}{\partial H'} \circ h^{-1})}(H))$ and $f_2(F) = (\frac{\partial g}{\partial F'} \circ g^{-1})(F)$, we obtain

$$\frac{\partial H}{\partial q_1} = -\frac{1}{G_2 f_1(H) f_2(F)} \frac{\partial F}{\partial q_1}. \tag{22}$$

We verify also that

$$\frac{\partial H}{\partial q_2} = -\frac{1}{G_1 f_1(H) f_2(F)} \frac{\partial F}{\partial q_2} \tag{23}$$

$$\frac{\partial H}{\partial p_1} = -\frac{1}{G_2 f_1(H) f_2(F)} \frac{\partial F}{\partial p_1} \tag{24}$$

$$\frac{\partial H}{\partial p_2} = -\frac{1}{G_1 f_1(H) f_2(F)} \frac{\partial F}{\partial p_2}. \tag{25}$$

According to the expressions (16) of C_0 and (19) of C, (21), (23)–(25) can be written:

$$C_0(., H) = -\frac{1}{G_1 G_2 f_1(H) f_2(F)} C(., F).$$

□

We now present an example illustrating the results achieved in proposition 3.

Example: Kolossof Hamiltonian [10]. It has been permitted to linearize the well known Kovalevskaya top.

We consider $M = \mathbb{R}^4$ with canonical coordinates (x, y, p_x, p_y), C_0 is the standard Poisson structure, and H, the square of Kolossof Hamiltonian given by

$$H = \left\{ \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{x^2 + y^2 - kx + 1}{\sqrt{x^2 + y^2}} \right\}^{\frac{1}{2}} \qquad k \in \mathbb{R}^*. \tag{26}$$

A direct calculation leads to the equation

$$C_0(.,H) = \frac{1}{-4k^2(x-k)^2 H\sqrt{F}} C(.,F) \tag{27}$$

where C is a second Poisson structure compatible with C_0, given by the following matrix

$$C = \begin{pmatrix} 0 & 0 & (x-k)^2 & (x-k)y \\ 0 & 0 & (x-k)y & y^2+k^2 \\ -(x-k)^2 & -(x-k)y & 0 & (x-k)p_y - yp_x \\ -(x-k)y & -(y^2+k^2) & -(x-k)p_y + yp_x & 0 \end{pmatrix} \tag{28}$$

and

$$F = \left\{ -\frac{1}{2}(k^2+y^2)p_x^2 + (x-k)yp_x p_y - \frac{1}{2}p_y^2(x-k)^2 - \frac{k(x-k)(kx-1)}{\sqrt{x^2+y^2}} \right\}^2. \tag{29}$$

So, $(\mathbb{R}^4, C_0, H)$ admits a QBHS with an integrating factor separable $\rho = -4k^2(x-k)^2 H\sqrt{F}$.

The eigenvalues λ_1 and λ_2 of the Nijenhuis operator $J = CC_0^{-1}$ satisfy

$$\lambda_1\lambda_2 = k^2(x-k)^2 \tag{30a}$$

$$\lambda_1 + \lambda_2 = (x-k)^2 + y^2 + k^2 \tag{30b}$$

and are functionally independent. So, we can find Nijenhuis coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$. The second part of the canonical transformation is given by

$$p_x = -2k\sqrt{\lambda_1\lambda_2}\left(\frac{p_{\lambda_1} - p_{\lambda_2}}{\lambda_1 - \lambda_2}\right) + 2\frac{\sqrt{\lambda_1\lambda_2}}{k}\left(\frac{\lambda_1 p_{\lambda_1} - \lambda_2 p_{\lambda_2}}{\lambda_1 - \lambda_2}\right) \tag{31a}$$

$$p_y = 2\frac{\lambda_1 p_{\lambda_1} - \lambda_2 p_{\lambda_2}}{\lambda_1 - \lambda_2}\sqrt{\lambda_1 + \lambda_2 - \frac{\lambda_1\lambda_2}{k^2} - k^2}. \tag{31b}$$

The relations (30a, b) and (31a, b) allow us to write H and F explicitly in the Nijenhuis coordinates:

$$H = \left\{ \frac{1}{\lambda_2 - \lambda_1}((2\lambda_2^2 - 2k^2\lambda_2)p_{\lambda_2}^2 + \lambda_2^{\frac{3}{2}} + (1-k^2)\lambda_2^{\frac{1}{2}} - ((2\lambda_1^2 - 2k^2\lambda_1)p_{\lambda_1}^2 + \lambda_1^{\frac{3}{2}} + (1-k^2)\lambda_1^{\frac{1}{2}})) \right\}^{\frac{1}{2}} \tag{32}$$

$$F = \left\{ \frac{1}{\lambda_2 - \lambda_1}(-\lambda_1((2\lambda_2^2 - 2k^2\lambda_2)p_{\lambda_2}^2 + \lambda_2^{\frac{3}{2}} + (1-k^2)\lambda_2^{\frac{1}{2}}) + \lambda_2((2\lambda_1^2 - 2k^2\lambda_1)p_{\lambda_1}^2 + \lambda_1^{\frac{3}{2}} + (1-k^2)\lambda_1^{\frac{1}{2}})) \right\}^2. \tag{33}$$

We see that

$$H = h\left(\frac{H_2(\lambda_2, p_{\lambda_2}) - H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right)$$

$$F = g\left(\frac{-\lambda_1 H_2(\lambda_2, p_{\lambda_2}) + \lambda_2 H_1(\lambda_1, p_{\lambda_1})}{\lambda_2 - \lambda_1}\right)$$

which imply Hamilton–Jacobi separability.

4. Some ideas for the general case

For the general case we must also obtain some relations between $LV(H)$, $LV(F)$ and $LV(f)$. The next proposition contains some indications for such a study in the future.

Proposition 5. Let $(M, C_0, H, C, F, \rho = -\lambda_1\lambda_2 f(H, F))$ be a QBHS. In terms of Nijenhuis coordinates, the following relations hold

$$LV(H) = -\frac{1}{\lambda_1^2\lambda_2 f^3} LV(F) \tag{34}$$

$$LV(f) = \left(\frac{\partial f}{\partial H} - \lambda_2 f \frac{\partial f}{\partial F}\right)\left(\frac{\partial f}{\partial H} - \lambda_1 f \frac{\partial f}{\partial F}\right)^2 LV(H) \tag{35}$$

(LV indicates the Levi-Civita operator (4)).

Proof. The proof is based on a direct calculation as follows.

(i) We get, in Nijenhuis coordinates, the value of the Levi-Civita operator applied to H: $LV(H)$; then we inject in $LV(H)$ the equations provided by the relation definition (1) of the QBHS (with $\rho = -\lambda_1\lambda_2 f$). By a grouping together of suitable terms, we obtain (34).

(ii) We search to identify $LV(H)$ or $LV(F)$ in $LV(f)$. Then using (34), we obtain (35). □

Remark 3. We observe that

$$\left(\frac{\partial f}{\partial H} - \lambda_i f \frac{\partial f}{\partial F}\right) = \frac{\frac{\partial H}{\partial \lambda_j}}{\frac{\partial f}{\partial \lambda_j}}$$

$i, j = 1, 2$ and $i \neq j$. So they do not vanish as $\frac{\partial H}{\partial \lambda_i} \neq 0$ and $\frac{\partial f}{\partial \lambda_i} \neq 0$ for $i = 1, 2$ (conditions always assumed for the Levi-Civita operator LV). Therefore we conclude that the study of the separability of the Hamilton–Jacobi equation associated with the Hamiltonian H can be subordinated to the value of $LV(f)$ one.

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